

# Quantum Action Principle for Covariant Systems. Bosonic string.

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A new form of covariant quantum theory based on a quantum version of the action principle is considered for the case of a free bosonic string. The central idea of the new approach is to delay conditions of stationarity of the classical action with respect to Lagrangian multipliers up to the quantum level where delayed conditions of stationarity are imposed on a quantum action. Physical states of the ordinary covariant quantum theory are replaced by well defined stationary states as those which obey quantum action principle. The stationary states have well defined energies.

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## I. INTRODUCTION

The present work is devoted to the development of a new approach to covariant quantum theory based on a quantum version of the action principle. A wide class of covariant theories includes gauge theories and precisely covariant theories with quadratic constraints on canonical momenta such as General Relativity and string theory. General canonical analysis and a variant of quantization of covariant theories was proposed by Dirac [1]. Classical action of a covariant theory has the canonical form:

$$I = \int d\tau \left( p_\alpha \dot{q}_\alpha - \lambda_a C_a \right), \quad (1)$$

where  $C_a$  are constraints, and  $\lambda_a$  are Lagrangian multipliers. Conditions of stationarity of the classical action (1) with respect to Lagrangian multipliers  $\lambda_a$  give the classical constraint equations,

$$C_a = 0. \quad (2)$$

In Dirac approach to covariant quantum theory, which may be called "ordinary" covariant approach, canonical variables  $q_\alpha$  and  $p_\alpha$  are replaced as usual by operators, but Schrödinger wave equation is replaced by the system of quantum constraints:

$$\hat{C}_a \psi = 0. \quad (3)$$

The set of wave equations (3) ensures the covariance at the quantum level. A solution  $\psi(q)$  of this set, being independent on a choice of Lagrangian multipliers  $\lambda_a$ , corresponds to a "physical state". It is not our goal to discuss all problems of the ordinary covariant quantum theory. The main problem is the problem of consistency of the system. The second problem is the problem of dynamical and probabilistic interpretation of physical states. Even in the simplest covariant quantum theory, i.e., one-particle relativistic quantum mechanics (RQM), we meet these problems [2].

A new approach to quantization of the dynamics of relativistic particle, which solves the problem of probabilistic interpretation of RQM, was proposed in [3, 4]. We call this new approach as a "quantum action principle (QAP)". In a general case, the main idea of QAP is to "delay" the conditions of the stationarity of the classical action with respect to Lagrangian multipliers  $\lambda_a$  up to the quantum level. At the quantum level in place of the system (3) we write the ordinary Schrödinger equation in the internal time parameter  $\tau$ :

$$i\hbar \frac{\partial \psi}{\partial \tau} = \lambda_a \hat{C}_a \psi. \quad (4)$$

Now the problem of the consistency of this set of equations is absent. A solution of the equation (4) defines an amplitude  $K \equiv \langle out | in \rangle$  for some quantum transition (for example, in real experiment)  $|in\rangle \rightarrow |out\rangle$ . Writing the transition amplitude in the exponential form,

$$K = \exp \left( \frac{i}{\hbar} \Lambda + R \right), \quad (5)$$

we consider its real phase function  $\Lambda$  as a quantum action, corresponding to the quantum transition  $|in\rangle \rightarrow |out\rangle$ . Being a functional of the Lagrangian multipliers  $\lambda_a$  (and kinematical parameters of the experiment), the quantum action  $\Lambda$  gives us a possibility to fix the Lagrangian multipliers  $\lambda_a$  by means of the "delayed" conditions of the stationarity:

$$\frac{\delta \Lambda}{\delta \lambda_a} = 0. \quad (6)$$

From Eq. (6) we obtain Lagrangian multipliers  $\lambda_a$  (more precisely, some invariants formed by  $\lambda_a$ ) as functions of kinematical parameters of the experiment. The only memory of quantum anomalies in our approach is the question: to what extent the covariance is conserved by the delayed conditions of the stationarity (6)? In any case, the transition amplitude (5) is well defined. Let us stress, that the equation (4) describes the dynamics of the system, in so far as kinematical parameters of boundary states  $|in\rangle$  and  $|out\rangle$  include the  $x^0$ -coordinate of the Minkowsky space.

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In the present work we develop the variant of QAP proposed in [4] in application to quantum theory of a closed bosonic string. In this approach, QAP is supplemented by a set of additional conditions which make the dynamics of the  $x^0$ -coordinate to be classical. This modification of QAP does not destroy the covariance of the theory, but it restores the explicit dynamical contents of the theory and solves the problem of probabilistic interpretation. In place of physical states defined by the set of equations (3) (if they exist) we propose a definition of stationary states as states which satisfy the principle of the stationarity of the quantum action  $\Lambda$ . In contrast to the ordinary definition of the spectrum of excitations of a string in fixed gauges (the light-cone gauge is the most wide used), our definition of stationary states is formally covariant. In the stationary states, the energy of a string takes certain values.

In the next section we will apply QAP to the well known simplest covariant object, i.e., a relativistic particle. Latter the new approach will be applied to a free bosonic string.

## II. QUANTUM ACTION PRINCIPLE IN RELATIVISTIC QUANTUM MECHANICS

We begin with the classical action of a relativistic particle in the canonical form

$$I = \int_0^1 d\tau \left( p_\mu \dot{x}^\mu - NH \right), \quad (7)$$

where

$$H \equiv p_\mu p^\mu - m^2 c^2 \approx 0 \quad (8)$$

is the Hamiltonian constraint equation, which is the Euler-Lagrange (EL) equation for the Lagrangian multiplier  $N$ . The action (7) is the reparameterization invariant. The constraint equation (8) is a restriction on the initial data in a phase space of a particle, and it does not define  $N$ . The "wavy" equality means, that the equation (8) has to be solved at the final stage, when all dynamical equations are taken into account. In fact, this means calculation of a stationary value of the classical action which may be performed in its Lagrangian form. If we take into account the EL equation for the momentum variable  $p_\mu$ ,

$$\dot{x}^\mu - 2Np^\mu = 0, \quad (9)$$

we return to the Lagrangian form of the classical action

$$I = \int_0^1 d\tau \left( \frac{\dot{x}^2}{4N} + m^2 c^2 N \right). \quad (10)$$

The action (10) also is a reparameterization invariant, but now the condition of its stationarity with respect

to the variable  $N$  fixes the reparameterization invariant integral:

$$T \equiv \int_0^1 d\tau N = \frac{\sqrt{(x_1 - x_0)^2}}{2mc}, \quad (11)$$

where  $x_{0,1}^\mu$  are end points of a world line of a particle. In this sense the Lagrangian form of the classical action is more fundamental, than canonical one [5]. The departure of the stationary conditions in two forms of the classical action will be traced in QAP which gives us a quantum analog of the Lagrangian form of the classical action. However, quantum mechanics is based on the canonical form of classical theory.

Let us turn to quantum mechanics. In the ordinary approach the classical constrain equation (8) is replaced by the Klein-Gordon (KG) equation

$$\left( \partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0. \quad (12)$$

The problem of interpretation of its solutions was mentioned above. Our proposal [3, 4] is to delay the condition of the stationarity with respect to the Lagrangian multiplier  $N$  up to formulation of a quantum dynamical problem. Following the logic of the work [4], let us slightly modify the original canonical action (7) as follows:

$$I = \int_0^1 d\tau \left[ p_\mu \dot{x}^\mu - N (d^2 - p_i^2 - m^2 c^2) - \lambda (d - p_0) \right]. \quad (13)$$

Additional variables  $d(\tau)$  and  $\lambda(\tau)$  might be excluded at the classical level with restoration of the original action (7). But we delay the restoration to the quantum level. This enlargement of the set of variational parameters will follow the classical character of the dynamics of the  $x^0$ -coordinate of a particle. In place of the KG equation (12) we write the modified Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = \left[ Nd^2 + \lambda \left( d - \frac{\hbar}{i} \frac{\partial}{\partial x^0} \right) - N (-\hbar^2 \Delta + m^2 c^2) \right] \psi. \quad (14)$$

Let  $\psi(\tau, x^\mu) = \psi_0(\tau, x^0) \psi_1(\tau, x^i)$ . The first multiplier, which describes quantum dynamics of the  $x^0$ -coordinate of a particle, obeys the equation

$$i\hbar \frac{\partial \psi_0}{\partial \tau} = \left[ Nd^2 + \lambda \left( d - \frac{\hbar}{i} \frac{\partial}{\partial x^0} \right) \right] \psi_0. \quad (15)$$

We look for a solution of the equation (15) in the exponential form:

$$\psi_0(\tau, x^0) = \exp \left[ \frac{i}{\hbar} \chi(\tau, x^0) \right], \quad (16)$$

where  $\chi(\tau, x^0)$  is a complex phase function for which a quadratic representation,

$$\chi(\tau, x^0) = \chi_0(\tau) + \chi_1(\tau)x^0 + \frac{1}{2}\chi_2(\tau)(x^0)^2, \quad (17)$$

is sufficient. We take  $\tau = 0$  at the initial moment,

$$\chi(0, x^0) = \frac{i\hbar}{4\varepsilon^2}(x^0)^2 + p_0x^0. \quad (18)$$

At the final stage of calculations the limit  $\varepsilon \rightarrow 0$  is supposed. It follows from (18), that the initial value of the  $x^0$ -coordinate lies in the  $\varepsilon$ -neighbourhood of zero, and corresponding initial momentum equals  $p_0$ . Comparing coefficients in both sides of the equation (14) in front of  $x^0$  degrees, we obtain:

$$\chi_2(\tau) = \chi_2(0) = \frac{i\hbar}{2\varepsilon^2}, \quad (19)$$

$$\chi_1(\tau) = p_0 + \frac{i\hbar}{2\varepsilon^2} \int_0^\tau d\tilde{\tau} \lambda(\tilde{\tau}), \quad (20)$$

$$\begin{aligned} \chi_0(\tau) = & - \int_0^\tau d\tilde{\tau} [Nd^2 + \lambda(d - p_0)] \\ & + \frac{i\hbar}{2\varepsilon^2} \int_0^\tau d\tilde{\tau} \int_0^{\tilde{\tau}} d\tilde{\tilde{\tau}} \lambda(\tilde{\tilde{\tau}}). \end{aligned} \quad (21)$$

The second part of a wave function which depends on spatial coordinates obeys the equation

$$i\hbar \frac{\partial \psi_1}{\partial \tau} = -N(-\hbar^2 \Delta + m^2 c^2) \psi_1. \quad (22)$$

In the present work we consider the problem of stationary states of relativistic systems. A corresponding solution of the equation (22) is a plane wave with the momentum  $p_i$ :

$$\begin{aligned} & \psi_1(\tau, x^i) \\ = & \exp \left[ \frac{i}{\hbar} \int_0^\tau d\tilde{\tau} N(\tilde{\tau}) (p_i^2 + m^2 c^2) - \frac{i}{\hbar} p_i x^i \right] \end{aligned} \quad (23)$$

Collecting together all parts of the stationary solution on the interval  $\tau \in [0, 1]$ , we obtain real functionals in the exponent (5):

$$\begin{aligned} \Lambda = & - \int_0^1 d\tilde{\tau} [Nd^2 + \lambda(d - p_0)] + p_0 \tilde{x}^0 \\ & + \int_0^1 d\tau N(p_i^2 + m^2 c^2) - p_i x^i, \end{aligned} \quad (24)$$

$$R = -\frac{1}{4\varepsilon^2} \left( \tilde{x}^0 + \int_0^1 d\tau \lambda \right)^2, \quad (25)$$

where  $\tilde{x}^0$  is a final value of the  $x^0$ -coordinate, corresponding to the state  $|out\rangle$ . In accordance with (25), the dynamics of  $x^0$ -coordinate is described by a wave packet with a width  $\varepsilon \rightarrow 0$ . Therefore, we have

$$\tilde{x}^0 + \int_0^1 d\tau \lambda = 0. \quad (26)$$

This equation must be added to the quantum action (24) as an additional condition.

Now we are ready to complete the formulation of QAP, solving the problem of the stationarity of the quantum action (24). Corresponding stationary equations for  $\lambda$  and  $d$  are

$$d = p_0, 2Nd + \lambda = 0, \quad (27)$$

and, in accordance with (26), we have a relation,

$$\tilde{x}^0 = 2Tp_0, T \equiv \int_0^1 d\tau N, \quad (28)$$

which is a quantum analog of the classical EL equation (9). Solving (28) with respect to the initial momentum  $p_0$ , and substituting this value in (24), we obtain:

$$\Lambda = \frac{(\tilde{x}^0)^2}{4T} + T(p_i^2 + m^2 c^2). \quad (29)$$

This is a quantum analog of the classical action in the Lagrangian form (10). The last step is calculation of a stationary value of (29) with respect to  $T$ ,

$$\Lambda = \tilde{x}^0 \sqrt{p_i^2 + m^2 c^2}. \quad (30)$$

Therefore, the solution of the problem of stationary states in RQM is the ordinary De-Broglie wave. In the next section we develop this approach in application to a quantum bosonic string.

### III. QUANTUM ACTION PRINCIPLE IN STRING THEORY

We consider here a closed bosonic string parameterized by two parameters  $(\tau, \sigma) \in [0, 1] \times [0, \pi]$  with classical action in the canonical form [6]:

$$I = \int_0^1 d\tau \int_0^\pi d\sigma \left[ p_\mu \dot{x}^\mu - N_1 H_1 - N_2 H_2 \right], \quad (31)$$

$$H_1 \equiv (p_\mu + \gamma x'_\mu)^2, H_2 \equiv (p_\mu - \gamma x'_\mu)^2, \quad (32)$$

where the dot denotes the derivative with respect to  $\tau$ , and the stroke denotes the derivative with respect to  $\sigma$ . The constraints (32) are in involution, i.e., their Poisson brackets (PB) equal zero. This is the consequence of covariance of the theory with respect to arbitrary transformations of coordinates  $(\tau, \sigma)$  on a world surface of a string. In order to guarantee the classical character of the dynamics of the  $x^0(\sigma)$ -coordinates of a string points at the quantum level, we enlarge the set of variational parameters in the classical action as follows:

$$I = \int_0^1 d\tau \int_0^\pi d\sigma \left\{ p_\mu \dot{x}^\mu - N_1 \left[ d_1^2 - (p_i + \gamma x'_i)^2 \right] - N_2 \left[ d_2^2 - (p_i - \gamma x'_i)^2 \right] - \lambda_1 \left[ d_1 - (p_0 + \gamma x'_0) \right] - \lambda_2 \left[ d_2 - (p_0 - \gamma x'_0) \right] \right\}. \quad (33)$$

Notice that the expressions in the square brackets under the integral are in involution because  $d_{1,2}$  have zero PB with any dynamical variable. Therefore, the covariance of the modified theory is not broken. This will help us to simplify the subsequent consideration, assuming that  $N_{1,2}$  are independent on  $\tau$ . Corresponding Schrödinger equation has the form

$$i\hbar \frac{\partial \psi}{\partial \tau} = \int_0^\pi d\sigma \left[ N_1 d_1^2 + N_2 d_2^2 + \lambda_1 d_1 + \lambda_2 d_2 - \lambda_1 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^0} + \gamma x'_0 \right) - \lambda_2 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^0} - \gamma x'_0 \right) - N_1 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^i} + \gamma x'_i \right)^2 - N_2 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^i} - \gamma x'_i \right)^2 \right] \psi. \quad (34)$$

We separate quantum dynamics of the  $x^0$ -coordinates and spatial coordinates of a string looking for a solution of the equation (34) as the product  $\psi[\tau, x^\mu(\sigma)] = \psi_0[\tau, x^0(\sigma)] \psi_1[\tau, x^i(\sigma)]$ . Quantum dynamics of the  $x^0$ -coordinates is described by an equation,

$$i\hbar \frac{\partial \psi_0}{\partial \tau} = \int_0^\pi d\sigma \left[ N_1 d_1^2 + N_2 d_2^2 + \lambda_1 d_1 + \lambda_2 d_2 - \lambda_1 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^0} + \gamma x'_0 \right) - \lambda_2 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^0} - \gamma x'_0 \right) \right] \psi_0. \quad (35)$$

We look for its solution in the exponential form

$$\psi_0[\tau, x^0(\sigma)] = \exp \left\{ \frac{i}{\hbar} \chi[\tau, x^0(\sigma)] \right\}, \quad (36)$$

$$\begin{aligned} \chi[\tau, x^0(\sigma)] &= \chi_0(\tau) + \int_0^\pi d\sigma \chi_1(\tau, \sigma) x^0(\sigma) \\ &\quad + \frac{1}{2} \int_0^\pi d\sigma \int_0^\pi d\tilde{\sigma} \chi_2(\tau, \sigma, \tilde{\sigma}) x^0(\sigma) x^0(\tilde{\sigma}) \end{aligned} \quad (37)$$

with symmetric third coefficient:  $\chi_2(\tau, \sigma, \tilde{\sigma}) = \chi_2(\tau, \tilde{\sigma}, \sigma)$ , and take at the initial moment  $\tau = 0$ :

$$\chi[\tau, x^0(\sigma)] = \frac{i\hbar}{4\varepsilon^2} \left( \int_0^\pi d\sigma (x^0)^2 \right)^2 + \int_0^\pi d\sigma p_0 x^0. \quad (38)$$

It means, that we take the initial value of  $x^0(\sigma)$  from the  $\varepsilon$ -neighbourhood of zero, and corresponding initial momentum equals  $p_0(\sigma)$ . Substituting (36) in the equation (35) and comparing coefficients in front of  $x^0$ -degrees in both sides, we obtain:

$$\chi_2(\tau, \sigma, \tilde{\sigma}) = \chi_2(0, \sigma, \tilde{\sigma}) = \frac{i\hbar}{2\varepsilon^2} \delta(\sigma - \tilde{\sigma}), \quad (39)$$

$$\begin{aligned} \chi_1(\tau, \sigma) &= -\gamma \int_0^\tau d\tilde{\tau} [\lambda'_1(\tilde{\tau}) - \lambda'_2(\tilde{\tau})] + p_0 \\ &\quad + \frac{i\hbar}{2\varepsilon^2} \int_0^\tau d\tilde{\tau} [\lambda_1(\tilde{\tau}) + \lambda_2(\tilde{\tau})], \end{aligned} \quad (40)$$

$$\chi_0(\tau) \quad (41)$$

$$\begin{aligned} &= - \int_0^\tau d\tilde{\tau} \int_0^\pi d\sigma (N_1 d_1^2 + N_2 d_2^2 + \lambda_1 d_1 + \lambda_2 d_2) \\ &\quad - \int_0^\tau d\tilde{\tau} \int_0^\pi d\sigma (\lambda_1 + \lambda_2) p_0 \\ &\quad - \gamma \int_0^\pi d\sigma \int_0^\tau d\tilde{\tau} (\lambda_1 + \lambda_2) \int_0^{\tilde{\tau}} d\tilde{\tau}' (\lambda'_1 - \lambda'_2) \\ &\quad + \frac{i\hbar}{2\varepsilon^2} \int_0^\pi d\sigma \int_0^\tau d\tilde{\tau} (\lambda_1 + \lambda_2) \int_0^{\tilde{\tau}} d\tilde{\tau}' (\lambda_1 + \lambda_2). \end{aligned}$$

We are ready to write the solution of the equation (35) on the interval  $\tau \in [0, 1]$  in terms of real functionals  $\Lambda_{x^0}, R_{x^0}$  defined by the exponential representation (5):

$$\begin{aligned} \Lambda_{x^0} &\quad (42) \\ &= - \int_0^1 d\tilde{\tau} \int_0^\pi d\sigma (N_1 d_1^2 + N_2 d_2^2 + \lambda_1 d_1 + \lambda_2 d_2) \\ &\quad + \int_0^1 d\tilde{\tau} \int_0^\pi d\sigma (\lambda_1 + \lambda_2) p_0 + \int_0^\pi d\sigma p_0 \tilde{x}^0 \\ &\quad - \gamma \int_0^\pi d\sigma \int_0^1 d\tilde{\tau} (\lambda_1 + \lambda_2) \int_0^{\tilde{\tau}} d\tilde{\tau}' (\lambda'_1 - \lambda'_2) \\ &\quad - \gamma \int_0^1 d\tilde{\tau} \int_0^\pi d\sigma (\lambda'_1 - \lambda'_2) \tilde{x}^0, \end{aligned}$$

$$R_{x^0} = -\frac{1}{4\varepsilon^2} \int_0^\pi d\sigma \left[ \tilde{x}^0 + \int_0^1 d\tilde{\tau} (\lambda_1 + \lambda_2) \right]^2, \quad (43)$$

where  $\tilde{x}^0(\sigma)$  is a final distribution of the  $x^0$ -coordinate along a string.  $\Lambda_{x^0}$  is a part of a quantum action corresponding to the dynamics of the  $x^0$ -coordinate of a string. Conditions of the stationarity of  $\Lambda_{x^0}$  with respect to additional variables  $d_{1,2}$  and  $\lambda_{1,2}$  are the following:

$$2N_{1,2}d_{1,2} + \lambda_{1,2} = 0, \quad (44)$$

$$\begin{aligned} 0 &= -d_1 + p_0 - \gamma \int_0^\tau d\tilde{\tau} (\lambda'_1 - \lambda'_2) \\ &\quad + \gamma \int_\tau^1 d\tilde{\tau} (\lambda'_1 + \lambda'_2) + (\tilde{x}^0)', \\ 0 &= -d_1 + p_0 - \gamma \int_0^\tau d\tilde{\tau} (\lambda'_1 - \lambda'_2) \\ &\quad - \gamma \int_\tau^1 d\tilde{\tau} (\lambda'_1 + \lambda'_2) - (\tilde{x}^0)' \end{aligned} \quad (45)$$

Taking derivatives of the equations (45) with respect to  $\tau$  and taking into account (44), we obtain

$$\frac{\partial}{\partial \tau} \left( \frac{\lambda_1}{2N_1} \right) = -2\gamma\lambda'_1, \quad \frac{\partial}{\partial \tau} \left( \frac{\lambda_2}{2N_2} \right) = 2\gamma\lambda'_2. \quad (46)$$

Taking  $\tau = 0$  in the equations (45), we obtain:

$$\lambda_{1,2}(0, \sigma) = \mp 2N_{1,2}(0, \sigma) p_0(\sigma). \quad (47)$$

Therefore, we have two wave equations (46) for  $\lambda_{1,2}$  - a wave to the right, and a wave to the left, and corresponding initial data (47) for both waves.

From the other hand, according to (43) in the limit  $\varepsilon \rightarrow 0$ , we have

$$\tilde{x}^0(\sigma) = - \int_0^1 d\tilde{\tau} (\lambda_1(\tau, \sigma) + \lambda_2(\tau, \sigma)). \quad (48)$$

The set of the equations (46), (47), and (48) defines  $p_0(\sigma)$  as a functional of  $N_{1,2}(\tau, \sigma)$ , and  $\tilde{x}^0(\sigma)$ . This functional is homogeneous of the +1 degree on  $x^0(\sigma)$ , and of the -1 degree on  $N_{1,2}(\tau, \sigma)$ . This result is an analog of the equation (28) in the case of a relativistic particle. Collecting all the results together, we obtain the stationary value of  $\Lambda_{x^0}$  (with respect to the additional variables  $d_{1,2}$  and  $\lambda_{1,2}$ ) as a functional, homogeneous of the +2 degree on  $x^0(\sigma)$ , and of the -1 degree on  $N_{1,2}(\tau, \sigma)$ .

Let us turn to the remained second multiplier of a wave function, which describes the dynamics of spatial coordi-

nates  $x^i$  of a string, and obeys the Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial \psi_1}{\partial \tau} \\ = - \left[ N_1 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^i} + \gamma x'_i \right)^2 + N_2 \left( \frac{\hbar}{i} \frac{\delta}{\delta x^i} - \gamma x'_i \right)^2 \right] \psi_1. \end{aligned} \quad (49)$$

Subsequent results will depend on a concrete physical problem under consideration. We are interested in stationary states of a quantum string. The Hamiltonian in the right hand side of the equation (49) describes a system of bound oscillators. Therefore, the stationary states we are looking for are excitations of these oscillators. However coefficients  $N_{1,2}(\sigma)$  in the Hamiltonian remain arbitrary up to now. For any eigenfunction of the Hamiltonian  $|n\rangle$ ,  $n = (n_1, n_2, \dots)$  with the energy  $E_n$ , the transition amplitude  $K_{nx^i} \equiv \langle out, n | in, n \rangle$  equals to

$$K_{nx^i} = \exp \left( -\frac{i}{\hbar} \int_0^1 d\tilde{\tau} E_n \right). \quad (50)$$

Notice, that the eigenvalues  $E_n$  of the Hamiltonian, as the Hamiltonian itself, are homogeneous functions of the first degree on  $N_{1,2}$ . Therefore, the same degree of homogeneity has the spatial part of a quantum action for stationary states:

$$\Lambda_{x^i} = - \int_0^1 d\tilde{\tau} E_n. \quad (51)$$

This result is analogous to the Eq.(23) in the case of a relativistic particle. Collecting both parts of the quantum action, we write it as the sum:

$$\Lambda = \Lambda_{x^0} + \Lambda_{x^i}. \quad (52)$$

This result is analogous to the Eq.(29) in the case of a relativistic particle. We get a quantum analog of the Lagrangian form of the classical action of a bosonic string.

Now, we are ready to complete the formulation of QAP by solving the conditions of the stationarity (6) with respect to remained arbitrary Lagrangian multipliers  $N_{1,2}$ . The structure of the action (52) is such, that stationary values of  $N_{1,2}$  will be homogeneous functionals of first degree on the  $\tilde{x}^0$ . Consequently, the stationary value of the action (52) itself will be a homogeneous functional of the first degree on the  $\tilde{x}^0$ . At last, we can take the boundary distribution  $\tilde{x}^0(\sigma) = \text{const}$  along a string. Then, the stationary value of the action (52) will be proportional  $\tilde{x}^0$ :

$$\Lambda = \tilde{x}^0 W_n, \quad (53)$$

where  $W_n$  is the energy of string in the stationary state  $|n\rangle$ . It is obvious, that the stationary value of the energy  $W_n$  is a homogeneous function of the 1/2-degree on quantum numbers  $n$ . Therefore, we have an analog of the square route in (30).

Is the stationary state  $|n\rangle$  in fact stationary? Yes, it is. For arbitrary values  $N_{1,2}(\sigma)$  the transition amplitude  $\langle out, m | in, n \rangle = 0$ , if  $m \neq n$ . This zero value will be conserved in the stationary point with respect to  $N_{1,2}(\sigma)$  (if it exists). Therefore, the probabilistic interpretation of the free string quantum dynamics in our framework is trivial, namely, a stationary state  $|n\rangle$  remains unchanged with the unit probability. Notice, that the stationary states  $|n\rangle$  are, in fact, non-orthogonal.

#### IV. CONCLUSIONS

In the present work we propose to impose the conditions of the stationarity of the classical action with respect to Lagrangian multipliers on to the quantum action. These conditions of the stationarity, i.e., constraints, are not to be solved at the classical level, or

transformed in quantum constraints, as in ordinary covariant quantum theory. We must form the transition amplitude of a real quantum process and take its real phase as a quantum action (of the process). The quantum action is an analog of the Lagrangian form of the classical action. The conditions of the stationarity of the quantum action with respect to the Lagrangian multipliers connect them (or more precisely, their invariant combinations) with kinematical parameters of the process. The only memory about quantum anomalies which arise in the ordinary covariant quantum theory is the question: what is the precision to which the Lagrangian multipliers are fixed in QAP? In any case, the transition amplitudes of quantum processes and stationary states are well defined in the Minkowsky space of arbitrary dimension.

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